Jour. Ind. Soc. Ag. Statistics
***43** (3), 1991 : 291-295*

Construction of Second Order Slope Rotatable Designs Through a Pair of Incomplete Block Designs

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(Received : June 1989)

Summary

A new method of constructing second order slope rotatable designs using two suitably chosen balanced incomplete block designs (BIBDs) is suggested. Slope rotatable central composite design (SRCCD) of Harder and Park [4] is shown to be obtainable using this method. It is observed that the method sometimes leads to designs with less number of design points than those available in the literature.

Key Words: Response surface designs, slope rotatability, second order slope rotatable designs.

Introduction

A second order response surface design $D = ((x_{iu}))$ for fitting

$$Y_u(x_{1u}, x_{2u}, x_{3u}, \ldots, x_{vu}) = b_o \sum_{i=1}^{v} b_{iu} x_{iu} + \sum_{i \le} \sum_{j} b_{ij} x_{iu} x_{ju} + e_u \qquad (1.1)$$

Where x_{iu} denotes the level of the i th factor (i=1, 2, . . ., v) in the uth run (u = 1, 2, . . ., N) of the experiment and e_u's are uncorrelated random errors with mean zero and variance σ^2, is said to be a second order slope rotatable design (SOSRD) if the variance of the estimate of first order partial derivative of $Y(x_1, x_2, \ldots, x_v)$ with respect to each of the independent variables (x_i) is only a function of the distance $d^2 \sum_{i}^{v} x_i^2$ of the point $(x_1, x_2, \ldots, x_v)$ from the origin (centre) of the design. Such a spherical variance function for estimation of slopes in second order surface is achieved if the design points satisfy the following conditions (see Harder and Park [4]).

1. $$\sum_{u=1}^{N} \prod_{i=1}^{v} x_{iu}^{\alpha_i} = 0 \text{ if any } \alpha_i \text{ is odd, for } \alpha_i = 0, 1, 2, 3$$
and $\alpha_i \leq 4$

2 .(i) $$\sum_{u=1}^{N} x_{iu}^2 = \text{constant} = N\lambda_2$$

(ii) $$\sum_{u=1}^{N} x_{iu}^4 = \text{constant} = cN\lambda_4$$

3. $$\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = \text{constant} = N\lambda_4$$

4. $$\frac{\lambda_4}{\lambda_2^2} > \frac{v}{c+v-1}$$

5. $$\lambda_4 \left[v(5-c)-(c-3)^2 \right] + \lambda_2^2 [v(c-5)+4] = 0 \quad (1.2)$$

where c, λ_2 and λ_4 are constants.

2. *New Method of Construction of SOSRD Through a Pair of BIBDs*

The method of construction of SOSRD through a pair of BIBDs, is given in the following Theorem (2.1). Here we use the notations of Das and Narasimham [3] and Narasimham *et al.* [5].

Theorem (2.1):

If $D_1 = (v, b_1, r_1, k_1, \lambda_1)$ and

$D_2 = (v, b_2, r_2, k_2, \lambda_2)$ are

two BIBDs, $2^{t(k_1)}$ and $2^{t(k_2)}$ denote Resolution V fractional replicates of 2^{k_1} and 2^{k_2} factorials with levels ± 1 and (n_o) is the number of central points, then the design points,

$$[1-(v, b_1, r_1, k_1, \lambda_1)] \times 2^{t(k_1)} \ U$$

$$[a-(v, b_2, r_2, k_2, \lambda_2)] \times 2^{t(k_2)} \ U(n_o)$$

give a v-dimensional SOSRD in

$$N = b_1 \times 2^{t(k_1)} + b_2 \times 2^{t(k_2)}\, U(n_o) \text{ design points,}$$

where a^2 is the positive real root of the biquadratic equation,

$$\left[2^{2t(k_2)} N \left\{\lambda_2^2 (5v-9) + r_2 \lambda_2 (6-v) - r_2^2\right\} + 2^{3t(k_2)} r_2^2 \left\{vr_2 - 5v\lambda_2 + 4\lambda_2\right\}\right] a^8 +$$

$$\left[2^{t(k_1)+2t(k_2)+1} r_1 r_2 (vr_2 - 5v\lambda_2 + 4\lambda_2)\right] a^6 +$$

$$\left[2^{t(k_1)+t(k_2)} N \left\{\lambda_1 \lambda_2 (10v-18) + (6-v)(r_1\lambda_2 + r_2\lambda_1) - 2r_1 r_2\right\} + \right.$$

$$\left. 2^{t(k_1)+2t(k_2)} r_2^2 (vr_1 - 5v\lambda_1 + 4\lambda_1) + 2^{2t(k_1)+t(k_2)} r_1^2 (vr_2 - 5v\lambda_2 + 4\lambda_2)\right] a^4 +$$

$$\left[2^{2t(k_1)+t(k_2)+1} r_1 r_2 (vr_1 - 5v\lambda_1 + 4\lambda_1)\right] a^2 +$$

$$\left[2^{2t(k_1)} N\left\{\lambda_1^2 (5v-9) + r_1 \lambda_1 (6-v) - r_1^2\right\}\right] +$$

$$\left[2^{3t(k_1)} r_1^2 (vr_1 - 5v\lambda_1 + 4\lambda_1)\right] = 0 \qquad (2.1)$$

If at least one positive real root exists for a^2 in equation (2.1) then the design exists. c can be determined from

$$c = \frac{r_1 \times 2^{t(k_1)} + r_2 \times 2^{t(k_2)} a^4}{\lambda_1 \times 2^{t(k_1)} + \lambda_2 \times 2^{t(k_2)} a^4} \qquad (2.2)$$

Proof: For the design points generated from the pair of BIBDs conditions (1), (2) and (3) of (1.2) are true. Condition (1) of (1.2) is true obviously. Conditions (2) and (3) of (1.2) are true a follows:

2. (i) $\sum x_{iu}^2 = r_1 \times 2^{t(k_1)} + r_2 \times 2^{t(k_2)} a^2 = N\lambda_2$

(ii) $\sum x_{iu}^4 = r_1 \times 2^{t(k_1)} + r_2 \times 2^{t(k_2)} a^4 = cN\lambda_4$

3. $\sum x_{iu}^2 x_{ju}^2 = \lambda_1 \times 2^{t(k_1)} + \lambda_2 \times 2^{t(k_2)} a^4 = N\lambda_4 \qquad (2.3)$

From (2) (ii) and (3) of (2.3), we get c given (2.2). Substituting for λ_2, λ_4 and c in (5) of (1.2) and on simplification, we get the biquadratic equation in a^2 given in (2.1).

Corollary (i): Taking D_1 = ($v = v$, $b_1 = 1$, $k_1 = v$, $r_1 = 1$, $\lambda_1 = 1$) and D_2 = (v=v, $b_2 = v$, $k_2 = 1$, $r_2 = 1$, $\lambda_2 = 1$), we get Hader and Park [4] SRCCD in $N = 2^{t(v)} + 2v + (n_o)$ design points as a particular case.

Corollary (ii): Taking D_1 = (v=v, $b_1 = b$, k_1= k, r_1= r, $\lambda_1 = \lambda$) and D_2 = (v=v, b_2=v, k_2=1, r_2=1, $\lambda_2 = 0$) in the above Theorem (2.1), we get a new method of construction of SOSRDs using a BIBD and star points.

We illustrate Theorem (2.1) with the construction of some SOSRDs.

Example (i): The design points,

$$[1- (v = 7,\ b_1= 7,\ r_1= 3,\ k_1= 3,\ \lambda_1= 1)] \times\ 2^3\ \ U$$

$$[a- (v = 7,\ b_2= 7,\ r_2= 1,\ k_2= 1,\ \lambda_2= 0)] \times\ 2^1\ \ U(n_o= 1)$$

will give a SOSRD in N=71 design points for 7-factors. This a new SOSRD.

Here (2.1) implies

$$228a^8 - 1344a^6 + 208a^4 + 7680a^2 - 17536 = 0 \qquad (2.4)$$

(2.4) has only one positive root namely $a^2 = 4.9748$.

Example (ii): The design points,

$$[1\text{-}(v=7,\ b_1=1,\ k_1=7,\ r_1=1,\ \lambda_1 = 1)] \times 2^6\ \ U$$

$$[a\text{-}(v=7,\ b_2=7,\ k_2=1,\ r_2=1,\ \lambda_2 = 0)] \times 2^1\ \ U\ (n_o = 1)$$

will give a SOSRD in 79 design points. Here (2.1) leads to a=3.7178. This is in fact the Hader and Park [4] SRCCD for 7-factors.

We note thus the new method some times leads to designs with lesser number of design points compared to SRCCDs.

ACKNOWLEDGEMENT

The authors are thankful to the referee and the editor for their constructive suggestions which have improved the earlier version of this paper.

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